You'll need these two integration techniques:

- Integration by substitution (setting $u = g(x)$):

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

For indefinite integrals,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where you find an antiderivative of $f(u)$ on the right side, and then substitute $u = g(x)$ to get back the answer in terms of x.

- Integration by parts (setting $u = f(x), v = g'(x)$):

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b g(x)f'(x)\,dx.$$

By "abuse of notation", we write it

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

For indefinite integrals, it becomes

$$\int u\,dv = uv - \int v du.$$

Problem 1
Compute the following indefinite integrals.

1. $\int \sin^2(x)\cos^3(x)\,dx$. *Hint:* $\cos^2(x) = 1 - \sin^2(x)$.
2. $\int \sin(x)\cos^3(x)\,dx$.
3. $\int \cos^2(x)\,dx$. *Hint: There is an easy way, and there is a hard way.*
4. $\int \sin^2(x)\cos^2(x)\,dx$.
5. $\int \log(x)\,dx$.
6. $\int x^n e^x\,dx$, for $n \in \mathbb{N}$.

Problem 2
For which $\alpha \in \mathbb{R}$ does the improper integral $\int_0^1 x^\alpha\,dx$ converge? What about $\int_1^\infty x^\alpha\,dx$?

Problem 3
[Question Redacted]

Suppose $f : [1, \infty) \to \mathbb{R}$ is decreasing, nonnegative, and $\displaystyle\int_1^\infty f(x)\,dx$ converges.

1. Show that for every $y \in (0, f(1)]$, there is a unique x such that $f(x) = y$.
2. Define $f^{-1} : (0, f(1)] \to \mathbb{R}$. Show that

$$\int_0^{f(1)} f^{-1}(y)\,dy = \int_1^\infty f(x)\,dx.$$

Hint: Draw graphs for both and compare the area under each curve.

Problem 4

1. Let $f : \mathbb{R} \to \mathbb{R}$ be an even, continuous function. Show that for any $a \in \mathbb{R}$ we have

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd, continuous function. Show that for any $a \in \mathbb{R}$ we have

$$\int_{-a}^{a} f(x)dx = 0$$

Remark: this is true for integrable functions in general, but a fun exercise in u-substitution.

Remark 2: It is tempting to look at the result of (2) and conclude that $\int_{-\infty}^{\infty} f(x)dx = 0$, but this is not the way we've defined the above integral; due to how chaotic things can get at infinity it's important the two infinities are considered separately.